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THE DOUBLE CENTRALIZER THEOREM FOR DIVISION ALGEBRAS

BY

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ABSTRACT

Assume V is a finite-dimensional vector space over a division ring D having center F. It is shown that if $T \in \mathrm{End}_D(V)$ is algebraic over F then the double centralizer $C(C(T))$ of T is the set $F[T]$ of all polynomials in T with coefficients from F. Consequently, each $n \times n$ matrix ring over D is an algebraic F-algebra if and only if $C(C(T)) = F[T]$ for all T and all finite-dimensional V.

Let D be a division ring with center F and let V be a finite dimensional (left) vector space over D. For any subset X of the algebra $\mathrm{End}_D(V)$ of D-endomorphisms of V, we denote the *centralizer* of X in $\mathrm{End}_D(V)$ by

$$C(X) = \{B \in \mathrm{End}_D(V) \mid AB = BA \text{ for all } A \in X\}.$$

In particular, for T in $\mathrm{End}_D(V)$ the centralizer $C(T)$ of T is the set of all D-endomorphisms of V which commute with T and the *double centralizer* $C(C(T))$ of T is the set of all D-endomorphisms which commute with every D-endomorphism which commutes with T.

When D is commutative, the double centralizer theorem, $C(C(T)) = F[T]$ for every D-endomorphism T of V, is a well-known classical theorem ([6] and [8, p. 106]).

In the noncommutative division ring case, R. Carlson and C. Cullen [2] have established this theorem when D is the real quarternion division algebra. Moreover, W. Werner [9] has extended this result to the case when D is a finite dimensional central division algebra.

But we note that the double centralizer theorem may not be true in the general division algebra case. In particular, if the vector space V is D itself and $C(C(T)) = F[T]$, T is necessarily algebraic over F, since T^{-1} is in $C(C(T))$.

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Thus if D has an element transcendental over its center F, then $C(C(T))$ properly contains $F[T]$ (compare with [7]).

However, in this paper we will obtain the double centralizer theorem in the case when the given D-endomorphism T of V is algebraic over F. Furthermore, as an application of this general result, we show the equivalence of the double centralizer property and the algebraicity of $\mathrm{Mat}_n(D)$, the $n \times n$ matrix algebra over D.

We start with the following well-known

LEMMA 1. *For a two-sided ideal I of $D[x]$ there exists a monic polynomial $f(x)$ in $F[x]$ such that $I = D[x]f(x)$. In particular, if I is prime, then $f(x)$ is irreducible in $F[x]$.*

PROOF. Since $D[x]$ is a principal (left and right) ideal domain, there is a monic polynomial $f(x)$ in $D[x]$ of least degree such that $I = D[x]f(x)$. For any $d \in D$, $r(x) = df(x) - f(x)d \in I$ and $\deg r(x) < \deg f(x)$. Hence $r(x) = 0$ and so $f(x) \in F[x]$. □

For a left ideal A of a ring R, $I(A) = \{r \in R \mid Ar \subseteq A\}$ is a subring of R and $I(A)$ is the largest subring of R in which A is a two-sided ideal. We call $I(A)$ the *idealizer* of A in R.

LEMMA 2. *Let R be a ring with identity and A be a left ideal. Then $\mathrm{End}_R(R/A) \cong I(A)/A$. That is, every R-endomorphism of R/A is a right multiplication by an element of $I(A)$.*

PROOF. See e.g. [3, p. 24]. □

Let $\mathrm{Ann}(M)$ denote the annihilator of M in $D[x]$, which is a two-sided ideal of $D[x]$.

THEOREM 3. *Let V be a finite dimensional left vector space over D. Then $C(C(T)) = F[T]$ for every $T \in \mathrm{End}_D(V)$ which is algebraic over F.*

PROOF. For $\sum d_i x^i \in D[x]$ and $v \in V$, define

$$\left(\sum d_i x^i\right) \cdot v = \sum d_i T^i(v).$$

Then V is a left $D[x]$-module and $\mathrm{Ann}(V) \neq 0$ since T is algebraic over F. Then $C(T) = \mathrm{End}_{D[x]}(V)$, hence, in order to calculate $C(C(T))$ we are actually determining the center of the ring $\mathrm{End}_{D[x]}(V)$. Now V can be expressed as a direct sum of cyclic $D[x]$-modules, $V \approx D[x]/q_1 \oplus \cdots \oplus D[x]/q_k$ where each is a left ideal of $D[x]$ and $q_1 \subseteq q_2 \subseteq \cdots \subseteq q_k$ [4, chapter 3]. By Lemma 2, each $D[x]$-endomorphism of $D[x]/q_i$ is given by a right multiplication by an element of $I(q_i)$ and $I(q_i) = D[x]$ only if q_i is an ideal of $D[x]$. Since it is not immediately evident that the q_i's can be chosen to be ideals of $D[x]$, we will replace V and T by a module W and linear transformation T_0 so that the cyclic decomposition of W is given by ideals of $D[x]$ and $C(C(T_0)) \approx C(C(T))$ as F-algebras. We proceed to the details.

Put $B = \mathrm{Ann}(V)$. Then by Lemma 1, there is a non-zero polynomial $f(x) \in F[x]$ such that $B = D[x]f(x)$.

Decompose $f(x) = p_1(x)^{m_1} p_2(x)^{m_2} \cdots p_n(x)^{m_n}$ into monic irreducible polynomials in $F[x]$, where $m_1, \cdots, m_n$ are positive integers.

Let

$$B_i = \mathrm{Ann}(p_i(x)^{m_i} D[x]),$$

and let

$$V^{(i)} = B_i V \qquad \text{for } i = 1, 2, 3, \cdots, n.$$

Then $V \cong \bigoplus \sum_{i=1}^{n} V^{(i)}$ as left $D[x]$-modules and $\mathrm{Ann}(V^{(i)}) = D[x]p_i(x)^{m_i}$. In particular, each $V^{(i)}$ is T-invariant.

For $i = 1, 2, \cdots, n$, we have a decomposition

$$V^{(i)} = W^{(i,1)} \oplus W^{(i,2)} \oplus \cdots \oplus W^{(i,k_i)}$$

into indecomposable cyclic $D[x]$-modules $W^{(i,j)}$, $1 \leqq j \leqq k_i$ and $\mathrm{Ann}(W^{(i,j)}) = D[x]p_i(x)^{e(i,j)}$ for suitable positive integers $e(i,j)$ with $m_i = e(i,1) \geqq e(i,2) \geqq \cdots \geqq e(i,k_i)$. Again, each $W^{(i,j)}$ is T-invariant.

By [4, theorem 20, p. 45], for $i = 1, \cdots, n$ we have a decomposition

$$\frac{D[x]}{D[x]p_i(x)^{e(i,j)}} = \bigoplus \sum \frac{D[x]}{D[x]q_{ij}(x)},$$

where the sum is finite and $D[x]/D[x]q_{ij}(x)$ is an indecomposable $D[x]$-module which is isomorphic to $W^{(i,j)}$ for $j = 1, \cdots, k_i$.

Now let

$$W = \bigoplus \sum_{i=1}^{n} \sum_{j=1}^{k_i} \frac{D[x]}{D[x]p_i(x)^{e(i,j)}}.$$

Thus W is a direct sum of modules U_{ij} and each U_{ij} is a direct sum of a finite number of copies of the indecomposable $D[x]$-module $W^{(i,j)}$. Hence the indecomposable direct summands of W and V coincide. We use T to define a

D-endomorphism T_0 on W as follows. Each U_{ij} is a direct sum of T-invariant subspaces. Thus T_0 on U_{ij} is the direct sum of T applied on each summand. In turn W is a direct sum of the U_{ij} so T_0 on W is the direct sum of T_0 on the U_{ij}. Then it may be easily checked that $f(T_0)\equiv 0$ on W, since $f(T)\equiv 0$ on V. Moreover, $F[T]\cong F[T_0]$ as F-algebras via the map which sends T to T_0 and fixes F.

For $\Sigma\, d_i x^i \in D[x]$ and $w\in W$, define an operation $\cdot$ by

$$\left(\sum d_i x^i\right)\cdot w=\sum d_i T_0^i(w).$$

Then this operation $\cdot$ is just the left $D[x]$-module operation on W, since the left $D[x]$-module V is just the $D[x]$-module $\bigoplus \Sigma\, D[x]/D[x]q_{ij}(x)$.

We rearrange summands of W so that

$$W=\bigoplus\sum_{i=1}^{l}\frac{D[x]}{D[x]q_i(x)},$$

where $q_i(x)\in F[x]$ and $D[x]q_1(x)\subseteq D[x]q_2(x)\subseteq\cdots\subseteq D[x]q_l(x)$. In fact $q_i(x)$ has the form $p_1(x)^{e(1,j_1)}p_2(x)^{e(2,j_2)}\cdots p_n(x)^{e(n,j_n)}$, where $e(k,j_k)$ may be possibly 0. Hence $q_i(x)\in F[x]$ since $p_j(x)\in F[x]$ for $j=1,\cdots,n$.

With this preparation, we will show that $C(C(T_0))=F[T_0]$. Suppose $g\in C(C(T_0))$. Considering W as the left $D[x]$-module, we have $C(T_0)=\mathrm{End}_{D[x]}(W)$. Since $g\in C(C(T_0))$, g commutes with that endomorphism of V which is the identity on $D[x]/D[x]q_1(x)$ and sends the other components to zero. Hence g induces a $D[x]$-endomorphism g_1 of $D[x]/D[x]q_1(x)$ and g_1 commutes with every $D[x]$-endomorphism of $D[x]/D[x]q_1(x)$. Since $D[x]q_1(x)$ is a *two-sided* ideal of $D[x]$, every $D[x]$-endomorphism of $D[x]/D[x]q_1(x)$ is a right multiplication by an element of $D[x]$ by Lemma 2. Hence g_1 is left multiplication by some $\alpha(x)\in D[x]$.

Let $h_{\alpha(x)}\colon W\to W$ be the left multiplication by $\alpha(x)$. Then by adopting G. Maxwell's technique in [7], $g\equiv h_{\alpha(x)}$ on W and g is a polynomial in T_0 with coefficients in F. Hence $C(C(T_0))=F[T_0]$.

Finally, we claim that $C(C(T))$ is F-algebra isomorphic to $C(C(T_0))$. Since $T\in C(T)$ and $T_0\in C(T_0)$, we have

$$C(C(T))=C_{\mathrm{End}_{D[x]}(V)}(C(T))=C_{\mathrm{End}_{D[x]}(V)}(\mathrm{End}_{D[x]}(V))$$

and

$$C(C(T_0))=C_{\mathrm{End}_{D[x]}(W)}(C(T_0))=C_{\mathrm{End}_{D[x]}(W)}(\mathrm{End}_{D[x]}(W)).$$

To define an F-algebra isomorphism from $C(C(T))$ to $C(C(T_0))$, recall that the list of $D[x]$-indecomposable summands of V and that of W coincide.

Let U be in the list of $D[x]$-indecomposable direct summands of W and Pr_U be the projection from W (or V) onto U. Define a map σ from $C(C(T))$ to $C(C(T_0))$ by the rule

$$\mathrm{Pr}_U \circ \sigma(h) = h \circ \mathrm{Pr}_U$$

for $h \in C(C(T))$. Then σ is an F-algebra isomorphism from $C(C(T))$ onto $C(C(T_0)) = F[T_0]$. Note that $\sigma(T^n) = T_0^n$ for any positive integer n, hence, σ induces an F-algebra isomorphism from $F[T] \subseteq C(C(T))$ onto $F[T_0]$. Therefore $F[T] = C(C(T))$ and the proof is completed. □

In [5] there is an interesting open question: If the division ring D is algebraic over F, then, is the matrix algebra $\mathrm{Mat}_n(D)$ algebraic for every n?

As an application of Theorem 3 we show that the double centralizer theorem is related to the above question.

THEOREM 4. *Let D be a division ring with center F. Then the following conditions are equivalent*:

(a) *For each positive integer n, the double centralizer property holds for all* $A \in \mathrm{Mat}_n(D)$.

(b) $\mathrm{Mat}_n(D)$ *is algebraic over F for each positive integer n.*

PROOF. By Theorem 3, (b) implies (a) immediately. Suppose (a) and let n be a positive integer. Take a left D-vector space V with dimension n. Then for any D-endomorphism T of V, there are T-invariant subspaces V_1 and V_2 of V such that $V = V_1 \oplus V_2$, $T_1 = T|_{V_1}$ is an isomorphism, $T_2 = T|_{V_2}$ is nilpotent by Fitting's lemma. Extend T_1 and T_2 to D-endomorphisms $\bar{T}_1$ and $\bar{T}_2$ of V, by stipulating that $\bar{T}_i = 0$ on V_{2-i} for $i = 1, 2$. Then $T = \bar{T}_1 + \bar{T}_2$ and $\bar{T}_1\bar{T}_2 = \bar{T}_2\bar{T}_1$.

By assumption (a), we have $C(C(T_1)) = F[T_1]$. But since T_1 is invertible, T_1 is algebraic over F. Obviously, T_2 is algebraic over F. Therefore $\bar{T}_1$ and $\bar{T}_2$ are algebraic over F. Now since $\bar{T}_1\bar{T}_2 = \bar{T}_2\bar{T}_1 = 0$, $T = \bar{T}_1 + \bar{T}_2$ is algebraic over F. □

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